# ON THE LEAST COMMON MULTIPLE OF POLYNOMIAL SEQUENCES AT PRIME ARGUMENTS

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ABSTRACT. Cilleruelo conjectured that if  $f \in \mathbb{Z}[x]$  is an irreducible polynomial of degree  $d \geq 2$  then,  $\log \text{lcm}\{f(n) \mid n < x\} \sim (d-1)x \log x$ . In this article, we investigate the analogue of prime arguments, namely,  $lcm{f(p) | p < x}$ , where  $p$  denotes a prime and obtain non-trivial lower bounds on it. Further, we also show some results regarding the greatest prime divisor of  $f(p)$ .

### 1. Introduction

For a polynomial  $f \in \mathbb{Z}[x]$ , define  $L_f(x) = \text{lcm}\{f(n) \mid n < x \text{ and } f(n) \neq 0\}$ , where the lcm of an empty set is taken to be 1. The Prime Number Theorem is equivalent to

$$
\log \operatorname{lcm}\{1, 2, \ldots, n\} \sim n.
$$

Therefore, we expect similar rate of growth for the case when f is a product of linear polynomials; see the article by Hong, Qian, and Tan [\[7\]](#page-8-0) for a thorough analysis of this case. However, the growth is not the same for higher degree polynomials. Cilleruelo in [\[2\]](#page-8-1) conjectured that  $\log L_f(x) \sim (d-1)x \log x$  for irreducible polynomials f of degree  $d > 2$  and proved it for  $d = 2$ . For some time,  $\log L_f(x) \gg x$  proven by Hong, Luo, Qian, and Wang in [\[6\]](#page-8-2), for polynomials with non-negative integer coefficients, was the strongest bound known. Recently, the conjectured order of growth was obtained by Maynard and Rudnick in [\[10\]](#page-9-0) and the bound was improved to  $x \log x$  by Sah in [\[12\]](#page-9-1). For a thorough survey on the least common multiple of polynomial sequences, see [\[1\]](#page-8-3).

In this article, we study the analogous problem at prime arguments. From the Prime Number Theorem, we know that

$$
\log \operatorname{lcm}\{p \mid p < x\} \sim x.
$$

This motivates us to consider  $\text{lcm}\{f(p) \mid p \lt x\}$  for an arbitrary polynomial  $f \in \mathbb{Z}[x]$ . For simplicity, we will only consider irreducible polynomials f.

<span id="page-0-0"></span>**Theorem 1.1.** Let  $f \in \mathbb{Z}[x]$  be an irreducible polynomial of degree d. Then,

 $\log \text{lcm}\lbrace f(p) \mid p < x \rbrace \gg x^{1-\varepsilon(d)},$ 

*where*  $\varepsilon(1) = 0.3735$ ,  $\varepsilon(2) = 0.153$  *and*  $\varepsilon(d) = \exp\left(\frac{-d - 0.9788}{2}\right)$  $\binom{0.9788}{2}$  *for*  $d \geq 3$ .

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We remark that  $\log \text{km}\lbrace f(p) \mid p < x \rbrace \leq (d + o(1))x \ll x$  follows from the Prime Number Theorem.

There is a lot of literature on the subject of largest prime divisor of  $p + a$  for some fixed integer a. Goldfeld in [\[4\]](#page-8-4) showed that there is a positive proportion of primes p such that  $p + a$  has a prime divisor greater than  $p^{\delta}$  for  $\delta = 0.5$ . The strongest known result in this regard is  $\delta = 0.677$  proven by Baker and Harman in [\[5,](#page-8-5) Theorem 8.3], an improvement of  $\delta = 0.6687$  obtained by Fouvry in [\[3\]](#page-8-6). Luca in [\[9\]](#page-9-2) obtained lower bounds on the proportion of such primes  $p$  for  $\delta \in [\frac{1}{4}]$  $\frac{1}{4}$ ,  $\frac{1}{2}$  $\frac{1}{2}$ . Similar work is also done for quadratic polynomials. Wu and Xi in [\[14\]](#page-9-3) proved that there exist infinitely many primes p such that  $p^2 + 1$  has a prime divisor greater than  $p^{0.847}$  by virtue of the Quadratic Brun-Titchmarsh theorem (see Theorem [2.4\)](#page-2-0) developed by the authors.

We obtain a result of a similar flavor for general polynomials which we state as follows.

<span id="page-1-1"></span>**Theorem 1.2.** Let  $f \in \mathbb{Z}[x]$  be an irreducible polynomial of degree d. Then, there *is a positive proportion of primes* p *such that* f(p) *has a prime divisor greater than*  $p^{1-\varepsilon(d)}$ , *where*  $\varepsilon(1) = 0.3735$ ,  $\varepsilon(2) = 0.153$  *and*  $\varepsilon(d) = \exp\left(\frac{-d - 0.9788}{2}\right)$  *for*  $d \geq 3$ .

The following table shows some values of  $1 - \varepsilon(d)$  for various d.

TABLE 1. Values of  $1 - \varepsilon(d)$ 

$\overline{1-\varepsilon(d)}$   0.6265   0.847   0.8632   0.9170   0.9496   0.9694   0.9814   0.9887				

**Notations.** We employ Landau-Bachmann notations  $\mathcal{O}$  and  $o$  as well as their associated Vinogradov notations ≪ and ≫. We say that  $a(x) \sim b(x)$  if

$$
\lim_{x \to \infty} \frac{a(x)}{b(x)} = 1.
$$

As usual, define  $\pi(x; m, a)$  to be the number of primes  $p \leq x$  such that  $p \equiv a$ a (mod m). Throughout the article, p and q will denote primes, and we fix an irreducible polynomial  $f \in \mathbb{Z}[x]$  of degree  $d > 1$ . We will often suppress the dependence of constants on  $f$ . At places, we may use Mertens' first theorem without commentary.

### 2. Background

<span id="page-1-0"></span>**Theorem 2.1** (Brun-Titchmarsh, [\[11\]](#page-9-4)). Let  $\theta = \frac{\log m}{\log r}$  $\frac{\log m}{\log x}$ , where  $\theta \in (0, 1)$ . Then,

$$
\pi(x; m, a) < (C(\theta) + o(1)) \cdot \frac{x}{\phi(m) \log x}
$$

*where*

$$
C(\theta) = \frac{2}{1 - \theta}.
$$

<span id="page-2-2"></span>Corollary 2.2. Let  $\varepsilon > 0$  be a constant. Then,

$$
\pi(x; m, a) \ll_{\varepsilon} \frac{x}{\phi(m) \log x}
$$

*for all positive integers*  $m < x^{1-\epsilon}$ .

<span id="page-2-4"></span>**Theorem 2.3** (Iwaniec, [\[8\]](#page-9-5)). Let  $\theta = \frac{\log m}{\log r}$  $\frac{\log m}{\log x}$  where  $\theta \in \left[\frac{9}{10}, \frac{2}{3}\right]$ 3 ]. *Then,*

$$
\pi(x; m, a) < (C(\theta) + o(1)) \cdot \frac{x}{\phi(m) \log x},
$$

*where*

$$
C(\theta) = \frac{8}{6 - 7\theta}.
$$

<span id="page-2-0"></span>**Theorem 2.4** (Wu and Xi, [\[15\]](#page-9-6)). Let  $A > 0$  and  $f(x)$  be an irreducible quadratic *polynomial. Define*  $\varsigma(m) = \#\{p < x \mid f(p) \equiv 0 \pmod{m}\}\$  *and*  $\rho(m)$  *to be the number of solutions of the congruence*  $f(x) \equiv 0 \pmod{m}$ . *For large*  $L = x^{\theta}$  *with*  $\theta \in [\frac{1}{2}]$  $(\frac{1}{2}, \frac{16}{17})$ , *we have* 

$$
\varsigma(m) \le (C(\theta) + o(1))\rho(m) \cdot \frac{x}{\phi(m)\log x},
$$

*for all*  $m \in [L, 2L]$  *with at most*  $\mathcal{O}_A(L/(\log L)^A)$  *exceptions, where* 

$$
C(\theta) = \begin{cases} \frac{124}{91 - 89\theta} & , \text{ if } \theta \in [\frac{1}{2}, \frac{64}{97}) \\ \frac{120}{86 - 83\theta} & , \text{ if } \theta \in [\frac{64}{97}, \frac{32}{41}) \\ \frac{28}{19 - 18\theta} & , \text{ if } \theta \in [\frac{32}{41}, \frac{16}{17}). \end{cases}
$$

<span id="page-2-1"></span>**Theorem 2.5** (Bombieri-Vinogradov). Let  $A \geq 6$  and  $Q \leq x^{\frac{1}{2}}/(\log x)^{A}$ . Then,

$$
\sum_{q\leq Q} \max_{2\leq y\leq x} \max_{(a,q)=1} \left| \pi(y;q,a) - \frac{y}{\phi(q)\log y} \right| \ll_A \frac{x}{(\log x)^B},
$$

*where*  $B = A - 5$ .

<span id="page-2-3"></span>**Lemma 2.6.** Let f be an irreducible integer polynomial and  $\rho(m)$  be the number *of roots of the congruence*  $f(x) \equiv 0 \pmod{m}$ . *Then,* 

$$
\sum_{p < x} \frac{\rho(p) \log p}{p - 1} = \log x + R + o(1)
$$

*for some constant* R.

*Proof.* By [\[13,](#page-9-7) 3.3.3.5], we have that

$$
\sum_{p < x} \rho(p) = \text{Li}(x) + \mathcal{O}\left(\frac{x}{(\log x)^3}\right),\,
$$

where  $Li(x)$  is the logarithmic integral. Applying Abel summation formula,

$$
\sum_{p < x} \frac{\rho(p) \log p}{p} = \frac{\log x}{x} \sum_{p < x} \rho(p) + \int_2^x \frac{\log x - 1}{x^2} \left( \sum_{p < u} \rho(p) \right) du + C_0
$$
\n
$$
= C_0 + 1 + \mathcal{O}\left(\frac{1}{\log x}\right) + \int_2^x \frac{\log u - 1}{u^2} \operatorname{Li}(u) du + \mathcal{O}\left(\int_2^x \frac{\log u - 1}{u(\log u)^3} du\right)
$$
\n
$$
= \log x + C_1 + \mathcal{O}\left(\frac{1}{\log x}\right)
$$

for some constants  $C_0$  and  $C_1$ . And the sum

$$
\sum_{p
$$

is  $C_2 + o(1)$  for some constant  $C_2$ . Hence, our lemma is proved.

## 3. Proof of Theorem [1.1](#page-0-0)

3.1. Setup. We study the product defined by

$$
Q(x) = \prod_{q < x} |f(q)| = \prod_{p} p^{\alpha_p(x)}
$$

and exploit the fact that the contribution of prime factors less than  $x^{\delta}$  is negligible compared to that of prime factors greater than  $x^{\delta}$ , where  $\delta$  is a parameter in  $(\frac{1}{2}, 1)$ to be chosen later. For some large enough constant B, set  $x_{\mathfrak{b}} = x^{1/2} (\log x)^{-B}$  for brevity.

Define  $\rho(m)$  to be the set of residues modulo m which satisfy the congruence  $f(x) \equiv 0 \pmod{m}$  and  $\rho(m)$  to be the cardinality of  $\rho(m)$ . Note that we have  $\rho(m) \leq d$  by Lagrange's theorem and that if  $p \nmid \text{disc } f$  then  $\rho(p) = \rho(p^n)$  for all  $n \geq 2$  by Hensel's lemma. Also define  $\varsigma(m)$  to be the sum

$$
\sum_{r \in \varrho(m)} \pi(x; m, r),
$$

the number of elements in  $\{f(p) | p < x\}$  divisible by m.

### 3.2. Estimating small primes. We define

$$
Q_S(x) = \prod_{p < x_{\mathfrak{b}}} p^{\alpha_p(x)},
$$

the part of  $Q(x)$  consisting of small prime divisors. The main result here is the following.

# <span id="page-3-0"></span>Proposition 3.1.  $\log Q_S(x) = \frac{x}{2} - \frac{Bx \log \log x}{\log x} + \mathcal{O}\left(\frac{x}{\log x}\right)$  $\frac{x}{\log x}\bigg)$

The proof uses an estimate on  $\alpha_p(x)$  making it easy to directly apply the Bombieri-Vinogradov theorem (Theorem [2.5\)](#page-2-1) in the end. The following result is proved by standard analysis involving Hensel's lemma and the Brun-Titchmarsh theorem (Corollary [2.2\)](#page-2-2).

<span id="page-4-0"></span>**Lemma 3.2.** *Let*  $p$  *be a prime. If*  $p \nmid \text{disc } f$ *, then* 

$$
\alpha_p(x) = \sum_{p^n < x_b} \varsigma(p^n) + \mathcal{O}\left(\frac{x}{\max\{p, x_b\} \log x} + \frac{(\log x)^{2B}}{\log p}\right);
$$

*else if* p | disc f, *we have*

$$
\alpha_p(x) = \varsigma(p).
$$

*Proof.* The case when  $p \mid \text{disc } f$  is easy to solve. So, let us assume  $p \nmid \text{disc } f$ . Observe that

$$
\alpha_p(x) = \sum_{n=1}^{\infty} \varsigma(p^n).
$$

When  $p^n \geq x$ , we see that  $\varsigma(p^n) \leq \rho(p^n) \leq d$ . If  $p^n$  divides  $f(k)$  for some  $1 \leq k \leq x$ , we have  $p^n \leq f(k) \leq f(x) < x^{d+1}$ , which implies that  $n < (d+1)\frac{\log x}{\log p}$ . Thus,

$$
\alpha_p(x) = \sum_{n=1}^{\infty} \varsigma(p^n) = \sum_{p^n < x} \varsigma(p^n) + \mathcal{O}\left(\frac{\log x}{\log p}\right).
$$

We split the summation into three intervals:  $p^{n} \in [1, x_{b}] \cup (x_{b}, x^{0.9}] \cup (x^{0.9}, x)$ . The last summation is

$$
\sum_{p^n \in (x^{0.9},x)} \varsigma(p^n) \le \sum_{p^n \in (x^{0.9},x)} \sum_{r \in \varrho(p^n)} \left( \frac{x}{p^n} + 1 \right) \le \sum_{p^n \in (x^{0.9},x)} \rho(p^n) (x^{0.1} + 1) \ll x^{0.2}.
$$

By Corollary [2.2,](#page-2-2) the second summation is

$$
\sum_{p^{n} \in (x_{b}, x^{0.9}] } \varsigma(p^{n}) \ll \frac{\rho(p)x}{\log x} \sum_{x_{b} < p^{n} \leq x^{0.9}} \frac{1}{\phi(p^{n})}
$$
\n
$$
\ll \frac{x}{\max\{p, x_{b}\} \log x} + \frac{x}{\log x} \sum_{\substack{n \geq 2 \\ x_{b} < p^{n} \leq x^{0.9}}} \frac{1}{p^{n}}
$$
\n
$$
\ll \frac{x}{\max\{p, x_{b}\} \log x} + \frac{x}{\log x} \cdot \frac{\log x}{\log p} \cdot \frac{1}{p^{2}}
$$
\n
$$
\ll \frac{x}{\max\{p, x_{b}\} \log x} + \frac{(\log x)^{2B}}{\log p}.
$$

Thus, our lemma is proved.

*Proof of Proposition [3.1.](#page-3-0)* Using Lemma [3.2,](#page-4-0)

$$
\log Q_S(x) = \sum_{p < x_b} \alpha_p(x) \log p
$$
\n
$$
= \sum_{p < x_b} \left( \sum_{p^n < x_b} \varsigma(p^n) + \mathcal{O}\left(\frac{x}{x_b \log x} + \frac{(\log x)^{2B}}{\log p}\right) \right) \log p
$$
\n
$$
= \sum_{m < x_b} \varsigma(m)\Lambda(m) + \mathcal{O}\left(\frac{x}{\log x}\right).
$$

Using Theorem [2.5](#page-2-1) and Lemma [2.6,](#page-2-3) we can estimate the above sum as

$$
\sum_{m < x_{\mathfrak{b}}} \varsigma(m) \Lambda(m) = \frac{x}{\log x} \sum_{m < x_{\mathfrak{b}}} \frac{\rho(m) \Lambda(m)}{\phi(m)} + \mathcal{O}\left(\frac{x}{(\log x)^{B-5}}\right)
$$
\n
$$
= \frac{x}{\log x} \left(\frac{1}{2} \log x - B \log \log x\right) + \mathcal{O}\left(\frac{x}{(\log x)^{B-5}}\right)
$$
\n
$$
= \frac{x}{2} - \frac{Bx \log \log x}{\log x} + \mathcal{O}\left(\frac{x}{\log x}\right),
$$

proving the result.  $\Box$ 

### 3.3. Removing medium-sized primes. Define the product

$$
Q_M(x) = \prod_{x_b \le p \le x^{1/2}} p^{\alpha_p(x)},
$$

the part of  $Q(x)$  consisting of medium-sized primes. The main result of this section is the following.

<span id="page-5-0"></span>Proposition 3.3.  $\log Q_M(x) \ll \frac{x \log \log x}{\log x}$ .

This means we can just remove medium-sized primes from  $\log Q(x)$  and only lose a sublinear portion. The proof is a simple computation using Lemma [3.2.](#page-4-0)

*Proof of Proposition [3.3.](#page-5-0)* From Lemma [3.2,](#page-4-0) it follows that

$$
\log Q_M(x) = \sum_{x_b \le p \le x^{1/2}} \alpha_p(x) \log p
$$
  
\n
$$
\ll \sum_{x_b \le p \le x^{1/2}} \left( \frac{x}{p \log x} + \frac{(\log x)^{2B}}{\log p} \right) \log p
$$
  
\n
$$
= \frac{x}{\log x} \sum_{x_b \le p \le x^{1/2}} \frac{\log p}{p} + \mathcal{O}(x^{1/2} (\log x)^{2B})
$$
  
\n
$$
\ll \frac{x \log \log x}{\log x},
$$

as desired.  $\Box$ 

### 3.4. Bounding large primes. Define the product

$$
Q_L(x) = \prod_{x^{1/2} < p < x^{\delta}} p^{\alpha_p(x)}
$$

,

the part of  $Q(x)$  consisting of large primes. The main result of this section is the following.

<span id="page-5-1"></span>Proposition 3.4.  $\log Q_L(x) \leq (1 + o(1))x \int_{1/2}^{\delta} C(\theta) d\theta$ .

The proof uses the Brun-Titchmarsh theorem (Theorem [2.1](#page-1-0) and [2.3\)](#page-2-4) and involves standard procedures to convert sums over primes to integrals.

*Proof of Proposition [3.4.](#page-5-1)* Let p be a prime in  $(x^{1/2}, x^{\delta})$ . Similar to the proof of Lemma [3.2,](#page-4-0) we have

$$
\alpha_p(x) = \sum_{n=1}^{\infty} \varsigma(p^n) = \varsigma(p) + \mathcal{O}(\log x/\log p) = \varsigma(p) + \mathcal{O}(1)
$$

as  $p^2 > x$ . Therefore,

$$
\log Q_L(x) = \sum_{x^{1/2} < p < x^{\delta}} \alpha_p(x) \log p
$$
\n
$$
= \sum_{x^{1/2} < p < x^{\delta}} \varsigma(p) \log p + O(x^{\delta}).
$$

By Theorem [2.1,](#page-1-0) [2.3](#page-2-4) and Lemma [2.6,](#page-2-3) we have

$$
\sum_{x^{1/2} < p < x^{\delta}} \varsigma(p) \log p \le \sum_{x^{1/2} < p < x^{\delta}} \frac{(C(\theta) + o(1))x}{\phi(p) \log x} \rho(p) \log p
$$
\n
$$
= \frac{x}{\log x} \sum_{x^{1/2} < p < x^{\delta}} \frac{C(\theta) + o(1)}{\phi(p)} \rho(p) \log p
$$
\n
$$
= \frac{x}{\log x} \sum_{x^{1/2} < p < x^{\delta}} C(\theta) \frac{\rho(p) \log p}{p - 1} + o\left(\frac{x \log \log x}{\log x}\right).
$$

It can be verified that the above inequality is true even when  $f$  is an irreducible quadratic polynomial and we apply Theorem [2.4](#page-2-0) instead of Theorem [2.3.](#page-2-4) By standard techniques to convert sums over primes into integrals, we have

$$
\sum_{x^{1/2} < p < x^{\delta}} \varsigma(p) \log p \le (1 + o(1))x \int_{1/2}^{\delta} C(\theta) \, \mathrm{d}\theta,
$$

proving the lemma.

3.5. The main bound. It is easy to see that

$$
\log Q(x) = \sum_{p < x} (d \log p + \mathcal{O}(1)) = dx + \mathcal{O}(x/\log x).
$$

Define

$$
Q_{VL}(x) = \prod_{p \ge x^{\delta}} p^{\alpha_p(x)},
$$

the part of  $Q(x)$  consisting of primes at least  $x^{\delta}$  (*very large* primes). Using Propositions [3.1,](#page-3-0) [3.3,](#page-5-0) and [3.4,](#page-5-1) we obtain

$$
\log Q_{VL}(x) = \log \frac{Q(x)}{Q_S(x)Q_M(x)Q_L(x)} \ge \left(d - \frac{1}{2} - \int_{1/2}^{\delta} C(\theta) \, \mathrm{d}\theta + o(1)\right)x.
$$

<span id="page-6-0"></span>Proposition 3.5.  $\log Q_{VL}(x) \geq \left(d - \frac{1}{2} - \int_{1/2}^{\delta} C(\theta) \; {\rm d}\theta + o(1) \right) x.$ 

3.6. **Bounding the integral.** The strategy will be to make  $\delta$  as large as possible while keeping Proposition [3.5](#page-6-0) non-trivial. Thanks to Theorem [2.1](#page-1-0) and [2.3,](#page-2-4) we are able to bound the integral effortlessly. For  $d \geq 2$ ,

$$
\int_{1/2}^{\delta} C(\theta) \, d\theta = \int_{1/2}^{2/3} C(\theta) \, d\theta + \int_{2/3}^{\delta} C(\theta) \, d\theta
$$
  
< 
$$
< \int_{1/2}^{2/3} \frac{8}{6 - 7\theta} \, d\theta + \int_{2/3}^{\delta} \frac{2}{1 - \theta} \, d\theta
$$
  
< 
$$
< -1.4788 - 2\log(1 - \delta).
$$

The case  $d = 1$  is a little special because we cannot make  $\delta$  greater than 2/3. For  $d=1,$ 

$$
\int_{1/2}^{\delta} C(\theta) \, d\theta < \int_{1/2}^{\delta} \frac{8}{6 - 7\theta} \, d\theta < 1.0472 - \frac{8}{7} \log(6 - 7\delta).
$$

3.7. Choosing  $\delta$ . To preserve the linear lower bound in Proposition [3.5,](#page-6-0) we want to have

$$
d - \frac{1}{2} \ge -1.4788 - 2\log(1 - \delta)
$$

if  $d \geq 2$ . This reduces to  $\delta \leq 1 - \exp\left(\frac{-d - 0.9788}{2}\right)$ . And for  $d = 1$ ,

$$
1 - \frac{1}{2} \ge 1.0472 - \frac{8}{7} \log(6 - 7\delta) \implies \delta \le 0.62656.
$$

However, we can do a lot better for  $d = 2$ , thanks to Theorem [2.4.](#page-2-0) The following numerical computation, also performed in [\[14\]](#page-9-3), shows that

$$
\int_{1/2}^{\delta} C(\theta) \, \mathrm{d}\theta \le \int_{\frac{1}{2}}^{\frac{64}{97}} \frac{124}{91 - 89\theta} \, \mathrm{d}\theta + \int_{\frac{64}{97}}^{\frac{32}{41}} \frac{120}{86 - 83\theta} \, \mathrm{d}\theta + \int_{\frac{32}{41}}^{\delta} \frac{28}{19 - 18\theta} \, \mathrm{d}\theta < \frac{3}{2}
$$

with  $\delta = 0.847$ . Thus, we set  $\delta = 1 - \varepsilon(d)$  for the rest of the argument, where  $\varepsilon(1) = 0.3735, \ \varepsilon(2) = 0.153, \text{ and } \varepsilon(d) = \exp\left(\frac{-d - 0.9788}{2}\right) \text{ for } d \geq 3.$ 

3.8. Finishing the argument. Define  $L(x) = \text{lcm}\lbrace f(p) \mid p < x \rbrace$ . Let p be a prime such that  $p \geq x^{\delta}$ . Note that the exponent of p in  $Q(x)$  is  $\mathcal{O}(x^{1-\delta})$ . We know that  $\log Q_{VL}(x) \gg x$ . Therefore,

$$
x \ll \log Q_{VL}(x) \ll x^{1-\delta} \sum_{\substack{p \ge x^{\delta} \\ p \mid Q(x)}} \log p.
$$

Thus,

$$
\log L(x) > \sum_{\substack{p \ge x^{\delta} \\ p | Q(x)}} \log p \gg x^{\delta},
$$

as desired.

**Remark 3.6.** It is worth noting that the same method gives  $\log \text{rad} \operatorname{cm} \{f(p) \}$  $p < x$ }  $\gg x^{1-\epsilon(d)}$ , similar to that obtained by Sah in [\[12\]](#page-9-1).

4. DIGRESSION ON THE GREATEST PRIME DIVISOR OF  $f(p)$ 

The main ingredient in proving Theorem [1.2](#page-1-1) is Proposition [3.5,](#page-6-0) which provides us a good handle on large primes dividing  $Q(x)$ .

*Proof of Theorem [1.2.](#page-1-1)* By Proposition [3.5,](#page-6-0)

$$
\log Q_{VL}(x) = \sum_{q < x} \sum_{\substack{p > x^{\delta} \\ p \mid f(q)}} \log p \gg x.
$$

Set  $\delta = 1 - \varepsilon(d)$ . Let the number of primes p less than x such that  $f(p)$  has a prime divisor greater than  $x^{\delta}$  be  $N(x)$ . Note that if  $p \mid Q(x)$ , then  $p < x^{d+1}$  for all large  $x$ . Thus,

$$
N(x) \gg \sum_{qx^{\delta} \\ p|f(q)}} 1 \gg \sum_{qx^{\delta} \\ p|f(q)}} \frac{\log p}{\log x^{d+1}} \gg \frac{1}{\log x} \sum_{qx^{\delta} \\ p|f(q)}} \log p \gg \frac{x}{\log x},
$$

which completes the proof.  $\Box$ 

Remark 4.1. *It can be seen that the Elliott–Halberstam conjecture allows us to take*  $\varepsilon$ (*d*) *to be any positive constant. For completeness, a formulation of the Elliott-Halberstam conjecture is as follows:*

Elliott-Halberstam Conjecture. *Define the error function*

$$
E(x;q) = \max_{\gcd(a,q)=1} \left| \pi(x;q,a) - \frac{\pi(x)}{\varphi(q)} \right|,
$$

*where the* max *is taken over all* a *relatively prime to* q. For every  $\theta < 1$  and  $A > 0$ , we have

$$
\sum_{1 \le q \le x^{\theta}} E(x; q) \ll_{\theta, A} \frac{x}{\log^{A} x}.
$$

We end the article with the following question for readers.

Question 4.2. Let f be an irreducible integer polynomial. Is it true that  $\log \text{cm}\{f(p) \mid$  $p < x \} \gg x?$ 

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