

ON QUOTIENTS OF VALUES OF EULER'S FUNCTION ON FACTORIALS

AYAN NATH AND ABHISHEK JHA

ABSTRACT. Recently, there has been some interest in values of arithmetical functions on members of special sequences, such as Euler's totient function φ on factorials, linear recurrences, etc. In this article, we investigate, for given positive integers a and b , the least positive integer $c = c(a, b)$ such that the quotient $\varphi(c!)/\varphi(a!)\varphi(b!)$ is an integer. We derive results on the limit of the ratio $c(a, b)/(a + b)$ as a and b tend to infinity. Furthermore, we show that $c(a, b) > a + b$ for all pairs of positive integers (a, b) with an exception of a set of density zero.

1. INTRODUCTION

In recent years, there has been some interest in values of arithmetical functions, especially Euler's totient function φ , on members of special sequences. Baczkowski et al. in [1] investigated arithmetic functions and factorials, precisely, $\varphi(n!), d(n!)$ and $\sigma(n!)$, where d is the divisor counting function and σ is summatory functions for divisors. In [6], Luca and Shparlinski obtained asymptotic formulas for moments of certain arithmetic functions with linear recurrence sequences. Further, Luca in [5] considered $\varphi(F_n)$, where F_n is the n th Fibonacci number. In [7], Luca and Stănică investigated quotients of the form $\varphi(C_m)/\varphi(C_n)$, where C_n is the n th Catalan number.

Luca and Stănică in [8] considered an analogue of binomial coefficients constructed by the Euler's totient function φ , which were previously proven to be integral by Edgar in [2]. The authors defined the φ -actorial as $n!_\varphi = \varphi(1)\varphi(2)\cdots\varphi(n)$, and the *phinomial* coefficient as

$$\binom{a+b}{a}_\varphi = \frac{(a+b)!_\varphi}{a!_\varphi b!_\varphi}.$$

In a similar spirit, we consider the quotient

$$\frac{\varphi((a+b)!)}{\varphi(a!)\varphi(b!)}.$$

Now, it does not take long to see that the above quantity is not always an integer. In fact, the quotient is not an integer for all pairs (a, b) with an exception of a set of density zero, as implied by Theorem 1.2, though there do exist arbitrarily large a and b such that the expression is an integer; indeed, it is an integer for $(a, b) = (n, \varphi(n!) - 1)$ (see Proposition 4.2 for more such pairs).

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Many interesting divisibilities in number theory are of the form $f(a)f(b) \mid f(c)$ where $f : \mathbb{N} \rightarrow \mathbb{N}$ is a function. A classical result of Erdős [3] states there is an absolute constant c so that if $a!b!$ divides $n!$ then $a + b < n + c \log n$, but for infinitely many values of n and some positive constant c , it follows that $n!/a!b!$ is an integer with $a + b = \lfloor n + c \log n \rfloor$. In fact, it is true that $(2n)!/(n!\lfloor n + c \log n \rfloor!)$ is an integer for all n with an exception of a set of density zero (see [4]). This motivates us to consider the quotient

$$\frac{\varphi(c!)}{\varphi(a!)\varphi(b!)}.$$

As $\varphi(n!)$ divides $\varphi(m!)$ whenever $m \geq n$, it makes sense to study the least positive integer c such that $\varphi(a!)\varphi(b!)$ divides $\varphi(c!)$. Let us denote the least such c for a pair (a, b) as $c(a, b)$. To understand the behaviour of $c(a, b)$ we plot a graph between $a + b$ and $c(a, b)$ for all pairs (a, b) with $1 \leq a, b \leq 100$. With the help of a computer, we obtain the plot shown in Figure 1.

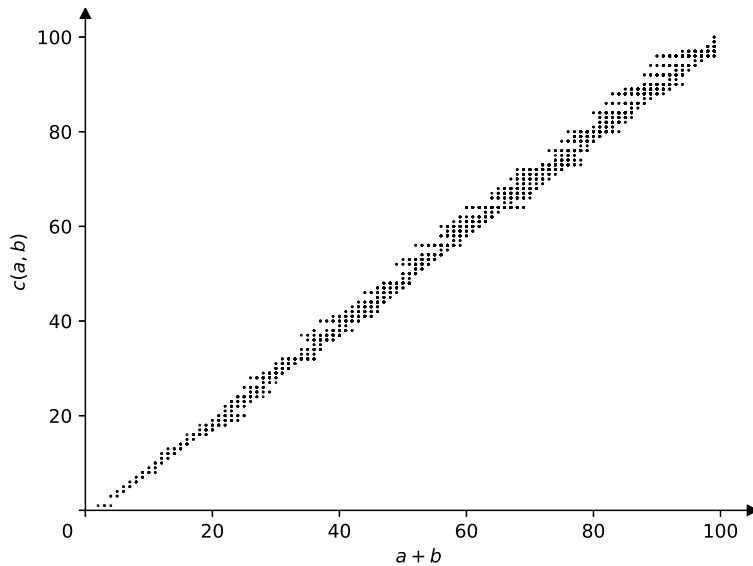


FIGURE 1. Plot of $c(a, b)$ versus $a + b$.

One immediately observes that the plot resembles the line $x = y$. This suggests that $c(a, b)$ is close to $a + b$, which motivates us to study the ratio $r(a, b) = c(a, b)/(a + b)$ for pairs (a, b) .

Definition 1.1. Define $c(a, b)$ as the least positive integer c such that $\varphi(a!)\varphi(b!)$ divides $\varphi(c!)$, and denote the ratio $c(a, b)/(a + b)$ as $r(a, b)$.

It is observed using a computer that almost always $r(a, b) > 1$. Table 1 shows the proportion of pairs (a, b) with $1 \leq a, b \leq N$ such that $r(a, b) > 1$ for various values of N .

The following theorem proved in this article confirms the evidence obtained in Table 1.

TABLE 1. Proportions of pairs (a, b) such that $r(a, b) > 1$.

N	100	200	300	400	500	600	700	800
Proportion	0.249	0.643	0.757	0.819	0.864	0.882	0.903	0.918

Theorem 1.2. *For all pairs of positive integers (a, b) , we have $r(a, b) > 1$ with an exception of a set of density zero.*

Seeing Figure 1, it is natural to ask whether $\lim_{a,b \rightarrow \infty} r(a, b)$ exists, and if yes, does it equal 1? Or, if the limit does not exist, what are the values of the limit inferior and the limit superior? The following theorem shows the exact value of the limit inferior.

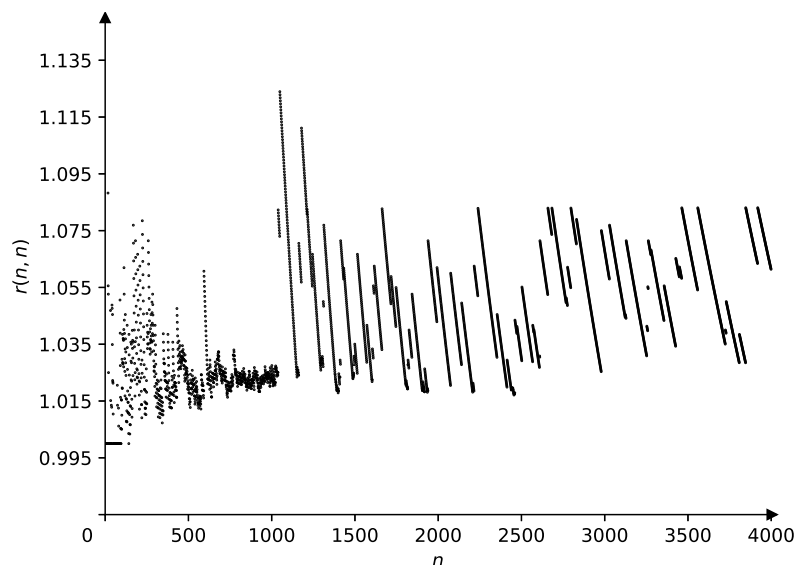
Theorem 1.3.

$$\liminf_{a,b \rightarrow \infty} r(a, b) = 1.$$

Studying the limit superior is equivalent to obtaining bounds on the ratio $r(a, b)$. We prove the following “sharp” upper bound on $r(a, b)$.

Theorem 1.4. *For all large positive integers a and b , we have $r(a, b) \leq \frac{9}{8}$.*

Unfortunately, it turns out that the sequence $r(a, b)$ fluctuates between values; it never stabilizes. Figure 2 shows a plot of n versus $r(n, n)$.

FIGURE 2. Fluctuation of $r(n, n)$.

Under the hypothesis of Dickson’s conjecture, a very well-believed and intuitive hypothesis in number theory, we prove the following theorem showing that the limit does not exist.

Theorem 1.5. *Assuming Dickson’s conjecture, there are infinitely many positive integers n such that*

$$r(n, n) \geq \frac{9}{8} - \frac{9}{8n}.$$

More importantly, Theorem 1.5 shows that Theorem 1.4 is sharp in the sense that the constant $9/8$ cannot be improved any further. It is now easy to see that Theorems 1.4 and 1.5 imply the following.

Corollary 1.6. *Assuming Dickson's conjecture, we have*

$$\limsup_{a,b \rightarrow \infty} r(a,b) = \frac{9}{8}.$$

The paper is organized as follows. In Section 2, we state some well-known results and prove a preliminary lemma. In Section 3, we show that almost all pairs (a,b) satisfy $r(a,b) > 1$ and hence prove Theorem 1.2. We present the proofs of Theorems 1.3 and 1.4 in Sections 4 and 5, respectively. Finally, in Section 6, we prove Theorem 1.5 under the hypothesis of Dickson's conjecture which concludes the result of Corollary 1.6.

Notations. We employ Landau-Bachmann notations \mathcal{O} and o as well as their associated Vinogradov notations \ll and \gg with their usual meanings. Throughout the article, the letters p and q are reserved for primes, and the letters a and b will always denote positive integers. As usual, define $\pi(x; m, a)$ to be the number of primes $p < x$ such that $p \equiv a \pmod{m}$. For a prime p and a non-zero integer n , define $\nu_p(n)$ as the exponent of p in the prime factorisation of n . For a non-zero rational number $r = a/b$ where a and b are integers, define $\nu_p(r) = \nu_p(a) - \nu_p(b)$ for any prime p .

2. PRELIMINARIES

Here we list out some classical results which are going to be helpful in our work. The following theorem is crucial in proving Lemma 2.4.

Theorem 2.1 (Siegel-Walfisz). *Let C be a positive constant. If a and q are two relatively prime positive integers such that $a < q \ll (\log x)^C$, then*

$$\left| \pi(x; q, a) - \frac{1}{\varphi(q)} \cdot \frac{x}{\log x} \right| \ll \frac{x}{(\log x)^B}$$

for some absolute constant $B > C + 1$.

Theorem 2.2 (Legendre). *For all positive integers n and primes p ,*

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n - s_p(n)}{p - 1},$$

where $s_p(n)$ is the sum of the digits of n in base- p .

Theorem 2.3 (Kummer). *Let p be a prime. The highest power of a prime p that divides $\binom{n}{k}$ is the number of carries in the addition $k + (n - k)$ in base- p .*

We know that $\varphi(n!) = n! \prod_{p \leq n} \frac{p-1}{p}$. So, to calculate the exponent of q in $\varphi(n!)$, we are interested in estimating $\nu_q(\prod_{p < x} (p-1))$, which is exactly what the following lemma is about.

Lemma 2.4. *If q is a fixed prime and K is a real constant, then*

$$\nu_q \left(\prod_{p < x} (p-1) \right) = \frac{q}{(q-1)^2} \cdot \frac{x}{\log x} + \mathcal{O} \left(\frac{x}{\log^K x} \right).$$

Proof. By a simple double counting argument, observe that

$$\nu_q \left(\prod_{p < x} (p-1) \right) = \sum_{q^n < x} \pi(x; q^n, 1).$$

We split the summation into two intervals: $1 \leq q^n < (\log x)^C$ and $(\log x)^C \leq q^n < x$. So,

$$\sum_{q^n < x} \pi(x; q^n, 1) = \sum_{q^n < (\log x)^C} \pi(x; q^n, 1) + \sum_{(\log x)^C \leq q^n < x} \pi(x; q^n, 1).$$

Using Theorem 2.1,

$$\begin{aligned} \sum_{q^n < (\log x)^C} \pi(x; q^n, 1) &= \sum_{q^n < (\log x)^C} \left(\frac{x}{\varphi(q^n) \log x} + \mathcal{O} \left(\frac{x}{\log^B x} \right) \right) \\ &= \frac{x}{\log x} \sum_{q^n < (\log x)^C} \frac{1}{q^{n-1}(q-1)} + \mathcal{O} \left(\frac{x \log \log x}{\log^B x} \right) \\ &= \frac{q}{(q-1)^2} \cdot \frac{x}{\log x} (1 + \mathcal{O}(\log^{-C} x)) + \mathcal{O} \left(\frac{x \log \log x}{\log^B x} \right) \\ &= \frac{q}{(q-1)^2} \cdot \frac{x}{\log x} + \mathcal{O} \left(\frac{x \log \log x}{\log^B x} \right), \end{aligned}$$

where B is the constant from Theorem 2.1. Also,

$$\sum_{(\log x)^C \leq q^n < x} \pi(x; q^n, 1) \leq \sum_{(\log x)^C \leq q^n < x} \frac{x}{q^n} = \mathcal{O} \left(\log x \frac{x}{(\log x)^C} \right) = \mathcal{O} \left(\frac{x}{\log^{C-1} x} \right).$$

Since $B > C + 1$, the lemma follows. \square

For convenience, let us define $\mathcal{T}(a, b; c)$ as follows.

Definition 2.5. *Define $\mathcal{T}(a, b; c)$ by*

$$\mathcal{T}(a, b; c) = \frac{\varphi(c!)}{\varphi(a!) \varphi(b!)}.$$

3. PAIRS (a, b) SUCH THAT $r(a, b) > 1$

In this section, we will prove Theorem 1.2.

Lemma 3.1. *Let A be any constant. If a and b are positive integers such that $b \geq a > \frac{b}{(\log b)^A}$, then $r(a, b) > 1$ for all sufficiently large b .*

Proof. Set $c = a + b$. We want to prove that $\mathcal{T}(a, b; c)$ is not an integer for all sufficiently large b . By routine manipulations, we obtain that

$$\mathcal{T}(a, b; c) = \binom{a+b}{a} \prod_{p \leq a} \frac{p}{p-1} \prod_{b < p \leq c} \frac{p-1}{p}.$$

We consider the largest power of 2 dividing $\mathcal{T}(a, b; c)$ and claim that it is negative for all large b . Using Theorem 2.3, we know that $\nu_2\left(\binom{a+b}{a}\right)$ is the number of carries in the base-2 addition of a and b , which is at most $1 + \log_2 b$. Therefore, by Lemma 2.4, we have

$$\begin{aligned} \nu_2(\mathcal{T}(a, b; c)) &= 2 \left(\frac{c}{\log c} - \frac{a}{\log a} - \frac{b}{\log b} \right) + \mathcal{O}\left(\frac{b}{\log^K b}\right) \\ &\ll \frac{2b}{\log^{K-1} b} + 2 \left(\frac{a+b}{\log(a+b)} - \frac{a}{\log a} - \frac{b}{\log b} \right) \\ &< 2 \left(\frac{b}{\log^{K-1} b} + \frac{a}{\log(a+b)} - \frac{a}{\log a} \right). \end{aligned}$$

Since $b/a < \log^A b$, the above expression is less than

$$2a \left(\frac{1}{\log^{K-1-A} b} + \frac{1}{\log(a+b)} - \frac{1}{\log a} \right).$$

It is routine to check that $\frac{1}{\log(a+b)} - \frac{1}{\log a}$ is increasing in $a \in (0, \infty)$. Hence, as $a \leq b$, the above expression is bounded above by

$$2a \left(\frac{1}{\log^{K-1-A} b} + \frac{1}{\log b + \log 2} - \frac{1}{\log b} \right) < -\frac{\delta a}{\log^2 b}$$

for some positive constant δ , provided $K - 1 - A > 2$, which can be ensured by taking K large. And the proof is complete. \square

With Lemma 3.1 in hand, it is easy to prove that the density of pairs (a, b) such that $r(a, b) > 1$ is 1. The proportion of pairs (a, b) such that $a = b$ is 0, so we can ignore them. It suffices to prove that the proportion of pairs (a, b) with $a < b$ and $r(a, b) > 1$ is $\frac{1}{2}$. Number of pairs (a, b) such that $1 \leq a < b \leq N$ and $r(a, b) > 1$ is at least

$$\begin{aligned} \sum_{b=N_0}^N \left((b-1) - \frac{b}{(\log b)^A} \right) &> \frac{N(N-1)}{2} - \frac{N^2}{(\log N)^A} + \mathcal{O}(1) \\ &= N^2 \left(\frac{1}{2} - \frac{1}{2N} - \frac{1}{(\log N)^A} + \mathcal{O}\left(\frac{1}{N^2}\right) \right), \end{aligned}$$

where N_0 is some constant only dependent on a and A . Thus, the density of such pairs is $\frac{1}{2}$, proving Theorem 1.2.

4. THE LIMIT INFERIOR

The proof of Theorem 1.3 has two steps. First, we need to prove that it is at least 1, and then show that there are arbitrarily large a and b such that $r(a, b) \leq 1$. The following result implies that the limit inferior is at least 1.

Lemma 4.1. *Let A be any real constant. If a and b are positive integers with $a \leq b$, then $c(a, b) \geq a + b - \frac{b}{\log^A b}$ for all sufficiently large b .*

Proof. Assume the contrary that $c = c(a, b) \leq a + b - \frac{b}{\log^A b}$ is true infinitely often. We calculate ν_2 of $\mathcal{T}(a, b; c)$ and claim that it is negative for all large b , which would contradict the assumption that $\mathcal{T}(a, b; c)$ is an integer. From Theorem 2.2, we have that $\nu_2(n!) = n + \mathcal{O}(\log n)$. Note that $b \leq c \leq a + b \leq 2b$. By Lemma 2.4,

$$\begin{aligned} \nu_2(\mathcal{T}(a, b; c)) &= \nu_2(c!) - \nu_2(a!) - \nu_2(b!) + 1 + \nu_2\left(\prod_{b < p \leq c} (p-1)\right) \\ &\quad - \nu_2\left(\prod_{p \leq a} (p-1)\right) \\ &= c - a - b + 2\left(\frac{c}{\log c} - \frac{a}{\log a} - \frac{b}{\log b}\right) + \mathcal{O}\left(\frac{b}{\log^K b}\right) \end{aligned}$$

for some constant K where we choose $K > A$. Note that

$$\begin{aligned} \frac{c}{\log c} - \frac{a}{\log a} - \frac{b}{\log b} &\leq \frac{c}{\log b} - \frac{a}{\log a} - \frac{b}{\log b} \\ &= \frac{c-b}{\log b} - \frac{a}{\log a} \\ &\leq \frac{c-b-a}{\log a} < 0. \end{aligned}$$

Therefore,

$$\nu_2(\mathcal{T}(a, b; c)) \leq -\frac{b}{\log^A b} + \mathcal{O}\left(\frac{b}{\log^K b}\right).$$

Since $K > A$, the above quantity is negative for all sufficiently large b , which is a contradiction to the supposition that $\mathcal{T}(a, b; c)$ is an integer. The proof is complete. \square

By Lemma 4.1, we see that

$$r(a, b) \geq 1 - \frac{b}{(a+b)\log^A b} \geq 1 - \frac{1}{2\log^A b}$$

for all sufficiently large b . It is now evident that

$$\liminf_{a, b \rightarrow \infty} r(a, b) \geq 1.$$

What remains now to be proven is that there exist arbitrarily large a and b such that $r(a, b) \leq 1$. Indeed, $\mathcal{T}(a, \varphi(a!) - 1; \varphi(a!)) = 1$ for all $a \geq 4$, so, $r(n, \varphi(n!) - 1) < 1$ for all $n \geq 4$. This completes the proof of Theorem 1.3.

However, we prove the following result, which says that, given a , many b satisfy $r(a, b) \leq 1$. The proof also demonstrates a natural way to construct such pairs.

Proposition 4.2. *Let a be a fixed positive integer. A positive proportion of positive integers b satisfy $r(a, b) \leq 1$.*

Proof. We provide a way to construct such b . At the end, it will be clear that we can ensure $b \geq a$, so let us assume $b \geq a$ from now onwards. Recall that

$$\mathcal{T}(a, b; a + b) = \binom{a + b}{a} \prod_{p \leq a} \frac{p}{p - 1} \prod_{b < p \leq a + b} \frac{p - 1}{p}.$$

Set $D = \prod_{p \leq a} (p - 1)$. Clearly, $\prod_{b < p \leq a + b} p$ divides $\binom{a + b}{a}$, and D is relatively prime to $\prod_{b < p \leq a + b} p$. Therefore, it suffices to prove that there exist infinitely many b such that D divides $\binom{a + b}{a}$. Let the prime factorisation of D be $\prod_{i=1}^m q_i^{\alpha_i}$. By Theorem 2.3, we want the addition of a and b in base- q_i to have at least α_i carries for each $i = 1, 2, \dots, m$. Therefore, we choose b such that $b \equiv -a \pmod{q_i^{\alpha_i}}$ for each $i = 1, 2, \dots, m$, and we take

$$b = k \prod_{i=1}^m q_i^{\alpha_i} - a = k \prod_{p \leq a} (p - 1) - a$$

for any positive integer k such that $b \geq a$. Thus, the proportion of positive integers b such that $r(a, b) \leq 1$ is positive. \square

Remark 4.3. *It can be noted that the size of b obtained in the above proof is around $\exp a(1 + o(1))$, and the proportion of such b is at least $\exp a(-1 + o(1))$.*

5. UPPER BOUND ON $r(a, b)$

In this section, we prove Theorem 1.4.

5.1. Setup. Set $c = a + b + \lfloor \frac{a+b}{8} \rfloor$. Without loss of generality, assume $a \leq b$. We wish to prove that $\mathcal{T}(a, b; c)$ is an integer for all large a and b . It is easy to see that

$$\mathcal{T}(a, b; c) = \binom{a + b}{a} (a + b + 1) \cdots (a + b + \lfloor \frac{a+b}{8} \rfloor) \prod_{p \leq a} \frac{p}{p - 1} \prod_{b < p \leq c} \frac{p - 1}{p}.$$

For brevity, call the above expression \mathcal{T} . We prove that \mathcal{T} is an integer by showing that $\nu_q(\mathcal{T}) \geq 0$ for all primes q . It is clear that $\nu_q(\mathcal{T}) \geq 0$ for all $q > a$. So, let us assume that $q \leq a$ for the rest of the argument. We split our proof into three cases: $q \leq 7$, $q \in [8, a^{1/2}]$ and $q \in (a^{1/2}, a]$. Take $a > 64$ so that the intervals do not overlap.

5.2. **Bounding $\nu_q(\mathcal{T})$ for $q \leq 7$.** This case is straightforward using Lemma 2.4. We have

$$\begin{aligned} \nu_q(\mathcal{T}) &\geq \nu_q((a+b+1) \cdots (a+b + \lfloor \frac{a+b}{8} \rfloor)) - \nu_q \left(\prod_{p \leq a} (p-1) \right) \\ &\geq \frac{1}{q} \left\lfloor \frac{a+b}{8} \right\rfloor + \mathcal{O} \left(\frac{a}{\log a} \right), \end{aligned}$$

which is positive for all large b as $a \leq b$.

5.3. **Bounding $\nu_q(\mathcal{T})$ for $q \in [8, a^{1/2}]$.** In this case, the estimates of Lemma 2.4 do not work because the error terms get large as q gets big. Hence, we require an alternative bound for $\nu_q \left(\prod_{p \leq a} (p-1) \right)$, which is stated as follows.

Lemma 5.1. *If $q > 7$, then*

$$\nu_q \left(\prod_{p \leq a} (p-1) \right) \leq \frac{0.23a}{q-1} + \frac{7 \log a}{\log q}.$$

To prove this, we use the following preliminary lemma.

Lemma 5.2. *Let $d > 7$ be a positive integer relatively prime to 3, 5 and 7. Then, the number of primes in $\{d+1, 2d+1, 3d+1, \dots, nd+1\}$ is at most $0.46n+7$.*

Proof. The proof is just a routine application of Inclusion-Exclusion principle. The number of elements in the set divisible by some positive integer k is either $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$. Applying Eratosthenes' sieve with respect to primes 3, 5 and 7, the number of primes in the set is at most

$$n - \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n}{5} \right\rfloor - \left\lfloor \frac{n}{7} \right\rfloor + \left\lceil \frac{n}{15} \right\rceil + \left\lceil \frac{n}{21} \right\rceil + \left\lceil \frac{n}{35} \right\rceil - \left\lfloor \frac{n}{105} \right\rfloor < 0.46n + 7,$$

as desired. \square

Proof of Lemma 5.1. Similar to the proof of Lemma 2.4, by a double counting argument and using Lemma 5.2,

$$\begin{aligned} \nu_q \left(\prod_{p \leq a} (p-1) \right) &= \sum_{2q^m < a} \pi(a; 2q^m, 1) \\ &\leq \sum_{2q^m < a} \left(\frac{0.46a}{2q^m} + 7 \right) \\ &< \frac{0.23a}{q-1} + \frac{7 \log a}{\log q}, \end{aligned}$$

as desired. \square

By a standard application of Theorem 2.2,

$$\begin{aligned} \nu_q((a+b+1) \cdots (a+b + \lfloor \frac{a+b}{8} \rfloor)) &\geq \nu_q(\lfloor \frac{a+b}{8} \rfloor!) \\ &\geq \frac{1}{q-1} \left\lfloor \frac{a+b}{8} \right\rfloor - \frac{\log(a+b)}{\log q} + \mathcal{O}\left(\frac{1}{\log q}\right) \\ &\geq \frac{a+b}{8(q-1)} - \frac{\log b}{\log q} + \mathcal{O}\left(\frac{1}{\log q}\right). \end{aligned}$$

Therefore,

$$\nu_q(\mathcal{T}) \geq \frac{a+b}{8(q-1)} - \frac{\log b}{\log q} + \mathcal{O}\left(\frac{1}{\log q}\right) - \left(\frac{0.23a}{q-1} + \frac{7 \log a}{\log q}\right).$$

Since $a \leq b$ and $q \leq a^{1/2}$, we have

$$\begin{aligned} \nu_q(\mathcal{T}) &\geq \frac{0.02a}{q-1} + \frac{b-a}{8(q-1)} - \frac{8 \log b}{\log q} + \mathcal{O}\left(\frac{1}{\log q}\right) \\ &\geq 0.02 \left(a^{1/2} + \frac{b-a}{400(q-1)} - \delta \log b \right) \end{aligned}$$

for some positive constants δ . Note that the above expression is non-negative if $a > \delta^2 \log^2 b$. Again, if $a < \delta^2 \log^2 b$, we see that $\frac{b-a}{400(q-1)}$ is greater than $\delta \log b$ for all sufficiently large b . Thus, $\nu_q(\mathcal{T}) \geq 0$ and this case is complete.

5.4. Bounding $\nu_q(\mathcal{T})$ for $q \in (a^{1/2}, a]$. Estimates like Lemma 2.4 and Lemma 5.1 are not suitable as the margin for slack is quite thin in this case. A careful analysis involving sharp bounds is required. Note that

$$\nu_q \left(\prod_{p \leq a} (p-1) \right) = \pi(a; 2q, 1).$$

So, we are interested in upper bounding $\pi(a; 2q, 1)$ sharply.

Lemma 5.3. *Let $q > 7$ be a prime. The number of primes in $\{2q+1, 4q+1, \dots, 2qk+1\}$ is at most $\lfloor \frac{k}{2} \rfloor + 1$, where equality can occur if $k = 4$ and $q = 11$.*

Proof. By Lemma 5.2, the number of primes in the set is at most $0.46k + 7$, which is less than or equal to $\lfloor \frac{k}{2} \rfloor + 1$ for all $k \geq 174$. We are going to use computational aids for the rest of k . Fix some $k \leq 173$. Consider the aforementioned set modulo $3 \cdot 5 \cdot 7 = 105$. Obviously, q can only be congruent to residues relatively prime to 105. We check all such residues one by one. A number in the set cannot be a prime if it is not relatively prime to 105. This procedure can be automated with a naive PARI/GP [9] script for all $k \leq 173$, completing the proof. \square

Similar to the previous cases, note that

$$\nu_q \left(\left\lfloor \frac{a+b}{8} \right\rfloor! \right) \geq \left\lfloor \frac{1}{q} \left\lfloor \frac{a+b}{8} \right\rfloor \right\rfloor \geq \left\lfloor \frac{a}{4q} \right\rfloor$$

as $a \leq b$. And by Lemma 5.3,

$$\pi(a; 2q, 1) \leq \left\lfloor \frac{1}{2} \left\lfloor \frac{a}{2q} \right\rfloor \right\rfloor + 1 = \left\lfloor \frac{a}{4q} \right\rfloor + 1,$$

where we use the basic fact that $\left\lfloor \frac{a}{4q} \right\rfloor = \left\lfloor \frac{1}{2} \left\lfloor \frac{a}{2q} \right\rfloor \right\rfloor$. Thus,

$$\begin{aligned} \nu_q(\mathcal{T}) &\geq \nu_q \left(\left\lfloor \frac{a+b}{8} \right\rfloor! \right) + \nu_q \left(\prod_{p \leq a} \frac{p}{p-1} \right) \\ &\geq \left\lfloor \frac{a}{4q} \right\rfloor + 1 - \left(\left\lfloor \frac{a}{4q} \right\rfloor + 1 \right) = 0, \end{aligned}$$

and the proof of Theorem 1.4 is complete.

6. DOES THE LIMIT EXIST?

Until now, we have proved results on the inferior and superior limits. A question that arises naturally is whether the limit exists. Numerical evidence suggests that the sequence $r(a, b)$ keeps on fluctuating (see Figure 2).

In this section, we provide a convincing argument for the non-convergence of the sequence using Dickson's conjecture and prove Theorem 1.5.

Hypothesis 6.1 (Dickson's Conjecture). *Let a_1, a_2, \dots, a_k be k integers and b_1, b_2, \dots, b_k be k positive integers. There are infinitely many positive integers n for which all the elements in the set $\{a_1 + b_1n, a_2 + b_2n, \dots, a_k + b_kn\}$ are primes, unless there is a congruence condition preventing this, i.e., there doesn't exist a prime q (necessarily at most k) such that the product $(a_1 + b_1n)(a_2 + b_2n) \cdots (a_k + b_kn)$ is divisible by q for all $n \in \{0, 1, \dots, q-1\}$.*

Let $q > 17$ be a sufficiently large prime such that $2q + 1$, $6q + 1$ and $8q + 1$ are primes but $iq + 1$ is not prime for each $i \in \{10, 12, 14, 16, 18\}$. Existence of infinitely many such primes q can be proven under Dickson's conjecture.

Lemma 6.2. *Assuming Dickson's conjecture, there exist infinitely many primes q such that $2q + 1, 6q + 1$ and $8q + 1$ are primes but $iq + 1$ is not prime for each $i = 10, 12, 14, 16, 18$.*

Proof. The main idea is to find one such prime q and apply Chinese Remainder Theorem. It is easy to see that q must be greater than 7. Observe that $q \equiv 2 \pmod{3}$ and $q \equiv 1 \pmod{5}$. Therefore, $iq + 1$ is divisible by 3 for each $i \in \{10, 16\}$, and $14q + 1$ is divisible by 5. It is easily verified that $q = 131$ satisfies the conditions of the statement. Indeed,

$$\begin{aligned} 2 \cdot 131 + 1 &= 263, & 6 \cdot 131 + 1 &= 787, & 8 \cdot 131 + 1 &= 1049, \\ 12 \cdot 131 + 1 &= 11^2 \cdot 13, & 18 \cdot 131 + 1 &= 7 \cdot 337. \end{aligned}$$

Consider the system of congruences

$$q \equiv 131 \pmod{11}, \quad q \equiv 131 \pmod{7}.$$

This is solved by $q \equiv 54 \pmod{77}$. So, if q is of the form $77\ell + 54$, then $12q + 1$ and $18q + 1$ are not primes. By Dickson's conjecture, there exist infinitely many ℓ such that $77\ell + 54, 2(77\ell + 54) + 1, 6(77\ell + 54) + 1$ and $8(77\ell + 54) + 1$ are primes. No modular conditions are preventing this because of the way we choose q . The proof is complete. \square

Take $n = 8q + 1$ and let $c(n, n) = 2n + m$ for some integer m . By Lemma 3.1, it is clear that m is positive. Let us assume for now that $m < \frac{n-9}{4}$; it will be clear later why we are assuming this. Recall that

$$\mathcal{T}(n, n; 2n + m) = \binom{2n}{n} (2n + 1)(2n + 2) \cdots (2n + m) \prod_{p \leq n} \frac{p}{p-1} \prod_{n < p \leq 2n+m} \frac{p-1}{p}.$$

We consider ν_q of the above expression. Since $q > 17$, and the base- q representation of n is 81, hence there are no carries in the addition $n + n$ in base- q . Therefore, by Theorem 2.3, q does not divide $\binom{2n}{n}$. Observe that

$$\nu_q \left(\prod_{p \leq n} \frac{p}{p-1} \right) \leq -2,$$

and, whether q divides $\prod_{n < p \leq 2n+m} (p-1)$ or not depends on how many of $9q+1, 10q+1, \dots, 18q+1$ are primes. Clearly, $9q+1, 11q+1, 13q+1, 15q+1$ and $17q+1$ are even. The remaining $iq+1$ for each $i \in \{10, 12, 14, 16, 18\}$ are not primes by assumption. Therefore, the product $\prod_{n < p \leq 2n+m} (p-1)$ has no contribution to the exponent of q . Lastly, the set $\{2n+1, 2n+2, \dots, 2n+m\}$ is equivalent to $\{3, 4, \dots, m+2\}$ when considered modulo q . Note that $2n + (q-2) = 17q$ is not divisible by q^2 . Therefore, for $\mathcal{T}(n, n; 2n+m)$ to be an integer, we must have $m+2 \geq 2q$, which implies that $m \geq \frac{n-9}{4}$. This is a contradiction to our initial assumption. Concluding,

$$c(n, n) \geq 2n + \frac{n-9}{4} = \frac{9n}{4} - \frac{9}{4}.$$

Thus,

$$r(n, n) \geq \frac{9}{8} - \frac{9}{8n},$$

completing the proof of Theorem 1.5.

Remark 6.3. *By a quantitative version of Dickson's conjecture, known in its very general form as Bateman-Horn conjecture, the number of integers in $[1, x]$ which satisfy the condition of Theorem 1.5 is of order at least $x/(\log x)^4$.*

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KALIABOR COLLEGE, KUWARITOL, ASSAM, INDIA

Email address: ayannath7744@gmail.com

INDRAPRASTHA INSTITUTE OF INFORMATION TECHNOLOGY, NEW DELHI, INDIA

Email address: abhishek20553@iiitd.ac.in